

Section 2.3 Product and Quotient Rules and Higher-Order Derivatives**THEOREM 2.7 The Product Rule**

The product of two differentiable functions  $f$  and  $g$  is itself differentiable. Moreover, the derivative of  $fg$  is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

**PROOF**

$$\begin{aligned} \frac{d}{dx}[f(x)g(x)] &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x)g(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x + \Delta x)g(x) + f(x + \Delta x)g(x) - f(x)g(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left[ f(x + \Delta x) \frac{g(x + \Delta x) - g(x)}{\Delta x} + g(x) \frac{f(x + \Delta x) - f(x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[ f(x + \Delta x) \frac{g(x + \Delta x) - g(x)}{\Delta x} \right] + \lim_{\Delta x \rightarrow 0} \left[ g(x) \frac{f(x + \Delta x) - f(x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} f(x + \Delta x) \cdot \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} g(x) \cdot \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= f(x)g'(x) + g(x)f'(x) \quad \blacksquare \end{aligned}$$

Note:

The Product Rule can be extended to cover products involving more than two factors. For example, if  $f$ ,  $g$ , and  $h$  are differentiable functions of  $x$ , then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

**Ex.1** Find the derivative of  $g(x) = (-7x + 9)(5x^3 - 4)$ .

Ex.2 Find the derivative of  $h(t) = \sqrt{t} \sin(t)$ .

### THEOREM 2.8 The Quotient Rule

The quotient  $f/g$  of two differentiable functions  $f$  and  $g$  is itself differentiable at all values of  $x$  for which  $g(x) \neq 0$ . Moreover, the derivative of  $f/g$  is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

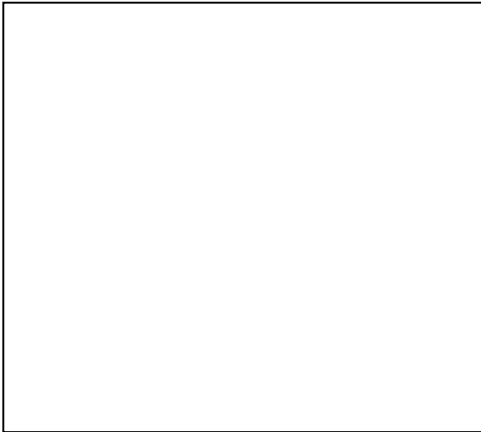
#### PROOF

$$\begin{aligned} \frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] &= \lim_{\Delta x \rightarrow 0} \frac{\frac{f(x + \Delta x)}{g(x + \Delta x)} - \frac{f(x)}{g(x)}}{\Delta x} && \text{Definition of derivative} \\ &= \lim_{\Delta x \rightarrow 0} \frac{g(x)f(x + \Delta x) - f(x)g(x + \Delta x)}{\Delta x g(x)g(x + \Delta x)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{g(x)f(x + \Delta x) - f(x)g(x) + f(x)g(x) - f(x)g(x + \Delta x)}{\Delta x g(x)g(x + \Delta x)} \\ &= \frac{\lim_{\Delta x \rightarrow 0} \frac{g(x)[f(x + \Delta x) - f(x)]}{\Delta x} - \lim_{\Delta x \rightarrow 0} \frac{f(x)[g(x + \Delta x) - g(x)]}{\Delta x}}{\lim_{\Delta x \rightarrow 0} [g(x)g(x + \Delta x)]} \\ &= \frac{g(x) \left[ \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \right] - f(x) \left[ \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \right]}{\lim_{\Delta x \rightarrow 0} [g(x)g(x + \Delta x)]} \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \quad \blacksquare \end{aligned}$$

**Ex.3** Find the derivative of  $g(t) = \frac{3t^2 - 6}{-5t + 4}$ .

**Ex.4** Find the derivative of  $f(a) = \frac{7\cos(a)}{6a^2}$ .

**Ex.4** Find the equation of the tangent line to the graph of  $f(x) = \frac{(x-1)}{(x+1)}$  at  $\left(2, \frac{1}{3}\right)$ .



**Ex.5** Use of the Constant Multiple Rule:

<u>Original Function</u>	<u>Rewrite</u>	<u>Differentiate</u>	<u>Simplify</u>
a. $y = \frac{x^2 + 3x}{6}$	$y = \frac{1}{6}(x^2 + 3x)$	$y' = \frac{1}{6}(2x + 3)$	$y' = \frac{2x + 3}{6}$
b. $y = \frac{5x^4}{8}$	$y = \frac{5}{8}x^4$	$y' = \frac{5}{8}(4x^3)$	$y' = \frac{5}{2}x^3$
c. $y = \frac{-3(3x - 2x^2)}{7x}$	$y = -\frac{3}{7}(3 - 2x)$	$y' = -\frac{3}{7}(-2)$	$y' = \frac{6}{7}$
d. $y = \frac{9}{5x^2}$	$y = \frac{9}{5}(x^{-2})$	$y' = \frac{9}{5}(-2x^{-3})$	$y' = -\frac{18}{5x^3}$

**THEOREM 2.9 Derivatives of Trigonometric Functions**

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$

**Ex.6** Find the derivatives of the following functions:

(a)  $\frac{d}{dx}[\tan(x)] =$

(b)  $\frac{d}{dx}[\sec(x)] =$

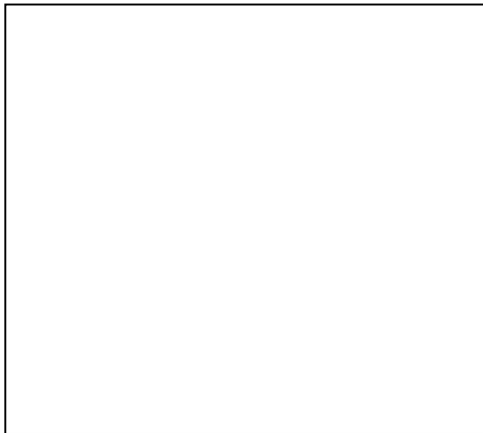
(c)  $\frac{d}{dx}[\cot(x)] =$

(d)  $\frac{d}{dx}[\csc(x)] =$

**Ex.7** Find the derivative of  $f(w) = \tan(w)\cot(w)$ .

**Ex.8** Find the derivative of  $h(\theta) = 5\theta\sec(\theta) + \theta\tan(\theta)$ .

**Ex.9** Find the equation of the tangent line to the graph of  $N(x) = \sec(x)$  at  $\left(\frac{\pi}{3}, 2\right)$ .



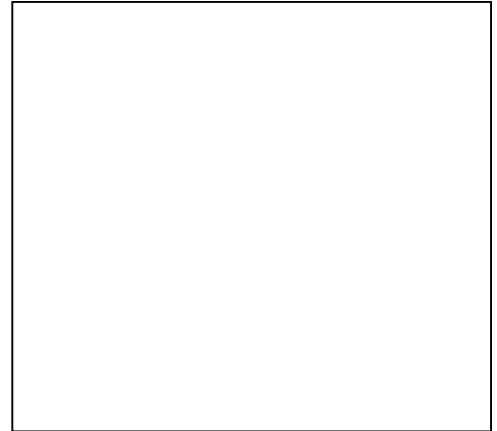
## Higher-Order Derivatives

**NOTE** The second derivative of  $f$  is the derivative of the first derivative of  $f$ .

**Ex.10** Find the second derivative of  $f(x) = 8x^6 - 10x^5 + 5x^3$ .

**Ex.11** Find the third derivative of  $f(x) = 2 - \frac{2}{x}$ .

**Ex.12** Determine the point(s) at which the graph of  $g(x) = \frac{x^2}{x^2+1}$  has a horizontal tangent line.





Ex.13 Given  $p(x) = f(x)g(x)$  and  $q(x) = \frac{f(x)}{g(x)}$ , use the graphs of  $f$  and  $g$  to find the following derivatives:

(a) Find  $p'(4)$ .

(b) Find  $q'(7)$ .

